Boolean valued models

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Structural results

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Boolean valued semantics for infinitary logic

Juan M Santiago

(joint work with Matteo Viale)

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- 3 Boolean valued models
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Let L be a signature. Formulas in first order logic are obtained by induction with the following rules:

- atomic formulas,
- $\blacksquare \ \phi \to \neg \phi \text{,}$
- $\label{eq:phi} \phi,\,\psi\to\phi\wedge\psi,\,\phi\vee\psi,$
- $\bullet \phi \to \exists v \phi, \forall v \phi.$

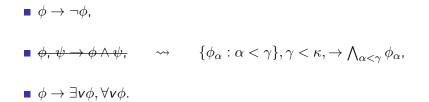
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Let κ be a regular cardinal, the rules of formation for the logic $L_{\kappa\omega}$ are:

atomic formulas,



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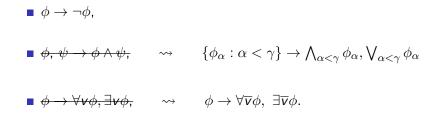
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Let $\lambda \leq \kappa$ be regular cardinals, the rules of formation for the logic $L_{\kappa\lambda}$ are:

atomic formulas,



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What can we say in $L_{\omega_1\omega}$?

Connected graph

$$\forall v \forall w \left(\bigvee_{n \in \omega} \exists v_1 \ldots \exists v_n (Rvv_1 \land \bigwedge_{i=1}^n Rv_i v_{i+1} \land Rv_n w) \right)$$



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Finitely generated group

$$\bigvee_{n\in\omega}\left(\exists v_1\ldots\exists v_n\forall x\bigvee_{\{i_1,\ldots,1_p\}\subset[1,n]}\bigvee_{\{m_1,\ldots,m_p\}\subset\omega}x=m_1v_{i_1}\ast\ldots\ast m_pv_{i_p}\right)$$

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What can we say in $L_{\omega_1\omega_1}$?

Well founded relation

$$\neg \left(\exists \{ v_n : n \in \omega \} \bigwedge_{n \in \omega} R v_{n+1} v_n \right)$$

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Well founded relation

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 $L_{\kappa\kappa}$ -elementary embeddings

$$j: V \to \mathrm{Ult}(V, G)$$

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Infinitary	logics
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But...

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But...

$$\mathbf{L} = \{ \boldsymbol{c}_n : n < \omega + 1 \} \cup \{ < \}$$

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But...

$$\mathbf{L} = \{c_n : n < \omega + 1\} \cup \{<\}$$

$$\mathbf{T} = \{ \forall x \bigvee_{n < \omega} (x = c_n) \} \cup \{ c_n < c_{n+1} : n \in \omega \} \cup \{ c_n < c_\omega : n < \omega \}$$

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Thus, compactness fails even for $L_{\omega_1\omega}$.

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How do we produce models of a sentence ϕ ?

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How do we produce models of a sentence ϕ ?

If compactness fails, lets try a Henkin construction:

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How do we produce models of a sentence ϕ ?

If compactness fails, lets try a Henkin construction:

•
$$\phi = orall v \psi(v) \quad \rightsquigarrow \quad \psi(c) ext{ for any } c \in C$$
 ,

$$\bullet \ \phi = \psi \lor \theta \quad \rightsquigarrow \quad \psi \text{ or } \theta.$$

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Definition

A consistency property is a set S of sets of $L_{\kappa\lambda}$ -formulas such that:

1 for any
$$\phi$$
, $\phi \notin s$ or $\neg \phi \notin s$,

2 if
$$\bigwedge \Phi \in s$$
, then $s \cup \{\phi\} \in S$ for any $\phi \in \Phi$,

3 if
$$\bigvee \Phi \in s$$
, then $s \cup \{\phi\} \in S$ for some $\phi \in \Phi$,

- 4 if $\forall \overline{v}\phi(\overline{v}) \in s$, then $s \cup \{\phi(\overline{c})\} \in S$ for any $\overline{c} \in C^{|\overline{v}|}$,
- 5 if $\exists \overline{v} \phi(\overline{v}) \in s$, then $s \cup \{\phi(\overline{c})\} \in S$ for some $\overline{c} \in C^{|\overline{v}|}$.

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Theorem (Model Existence Theorem, Makkai 1969)

Let L be a countable signature. If S is a consistency property for $L_{\omega_1\omega}$ whose elements are all countable, then any $s \in S$ is consistent.

Corollary

A sentence $\phi \in L_{\omega_1\omega}$ has a model if and only if there exists a consistency propety S and some $s \in S$ with $\phi \in s$.

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Forcing with consistency properties

Definition

Let S be a consistency property for $L_{\kappa\omega}$. Define the forcing notion $\mathbb{P}_S = (P, \leq)$: P = S, s < t if and only if $t \subset s$.

Note that in this context the conditions in the definition of a consistency property talk about density. For example:

$$\forall v \phi(v) \in s$$
, then $s \cup \{\phi(c)\} \in S$ for any $c \in C$,

is telling us that the sets of the form $D_{\phi(c)} = \{t \in S : \phi(c) \in t\}, c \in C$, are dense below the condition s.

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Theorem (Model Existence Theorem, S. & Viale)

Let S be a consistency property for $L_{\kappa\omega}$. If $s \in S$ and G is a V-generic filter containing s, then s is consistent in V[G].

Corollary

A sentence $\phi \in L_{\kappa\omega}$ has a model in some generic extension if and only if for some consistency propety S in V and some $s \in S$, $\phi \in s$.

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Generalizing Tarski semantics

Instead of thinking about truth as a 1/0 function, lets take a more general perspective.

Let B be a boolean algebra, the truth value of a sentence in a B-valued model ${\mathcal M}$ is a function

 $\phi \mapsto [\phi]_{\mathsf{B}}^{\mathcal{M}} \in \mathsf{B}.$

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Generalizing Tarski semantics

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$$\phi \mapsto [\phi]_{\mathsf{B}}^{\mathcal{M}} \in \mathsf{B}.$$

$$[\neg \phi]_{\mathsf{B}}^{\mathcal{M}} = \neg [\phi]_{\mathsf{B}}^{\mathcal{M}}$$

$$[\bigvee \Phi]_{\mathsf{B}}^{\mathcal{M}} = \bigvee_{\phi \in \Phi} [\phi]_{\mathsf{B}}^{\mathcal{M}} \qquad [\bigwedge \Phi]_{\mathsf{B}}^{\mathcal{M}} = \bigwedge_{\phi \in \Phi} [\phi]_{\mathsf{B}}^{\mathcal{M}}$$

$$[\exists v \phi(v)]_{\mathsf{B}}^{\mathcal{M}} = \bigvee_{m \in \mathcal{M}} [\phi(m)]_{\mathsf{B}}^{\mathcal{M}} \qquad [\forall v \phi(v)]_{\mathsf{B}}^{\mathcal{M}} = \bigwedge_{m \in \mathcal{M}} [\phi(m)]_{\mathsf{B}}^{\mathcal{M}}$$

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Full boolean valued models

It might happen that $\ensuremath{\mathcal{M}}$ is a B-valued model realizing an existential sentence,

$$[\exists \mathbf{v}\phi(\mathbf{v})]_{\mathsf{B}}^{\mathcal{M}}=\mathbf{1}_{\mathsf{B}},$$

but for no $m \in M$,

 $[\phi(m)]_{\mathsf{B}}^{\mathcal{M}} = 1_{\mathsf{B}}.$

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Full boolean valued models

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$$[\phi(m)]_{\mathsf{B}}^{\mathcal{M}}=1_{\mathsf{B}}.$$

Definition

A B-valued model \mathcal{M} is full if for every existential formula $\exists v \phi(v)$ there exists some $m \in M$ such that

$$[\exists v\phi(v)]_{\mathsf{B}}^{\mathcal{M}} = [\phi(m)]_{\mathsf{B}}^{\mathcal{M}}.$$

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Model existence theorems

Theorem (Model Existence Theorem, Mansfield 1972)

Let S be a consistency property for $L_{\kappa\kappa}$. For any $s \in S$ there exists \mathcal{M} a boolean valued model realizing s,

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Theorem (Model Existence Theorem, Viale & S.)

Let S be a consistency property for $L_{\kappa\omega}$. For any $s \in S$ there exists \mathcal{M} a full boolean valued model realizing s,

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Completeness

Theorem (Boolean Completeness for $L_{\kappa\kappa}$, Mansfield)

The following are equivalent for T, S sets of $L_{\kappa\kappa}$ -formulae.

- 1 $T \models_{\text{BVM}} S$,
- 2 $T \vdash S$.

Theorem (Boolean Completeness for $L_{\kappa\omega}$, S. & Viale)

The following are equivalent for T, S sets of $L_{\kappa\omega}$ -formulae.

1
$$T \models_{\text{Sh}} S$$
,
2 $T \models_{\text{BVM}} S$

3
$$T \vdash S$$
.

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Craig's interpolation

Theorem (Boolean Craig Interpolation for ${ m L}_{\kappa\lambda})$

Assume $\vDash_{BVM} \phi \rightarrow \psi$ with $\phi, \psi \in L_{\kappa\lambda}$. Then there exists a sentence θ in $L_{\kappa\lambda}$ such that

- $\blacksquare \vDash_{\mathrm{BVM}} \phi \to \theta, \vDash_{\mathrm{BVM}} \theta \to \psi,$
- all non logical symbols appearing in θ appear both in ϕ and ψ .

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Omitting types theorems

Theorem (Boolean Omitting Types Theorem for $L_{\kappa\lambda}$, S. & Viale)

Let T be a boolean satisfiable $L_{\kappa\lambda}$ -theory. Let $\Sigma(\overline{\nu})$ be an $L_{\kappa\lambda}$ type and assume that it is not isolated by an $L_{\kappa\lambda}$ -sentence modulo T. Then there exists a boolean valued model \mathcal{M} such that:

• $\mathcal{M} \vDash T$ and \mathcal{M} omits Σ .

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Theorem (Boolean Omitting Types Theorem for $L_{\kappa\omega}$, S. & Viale)

Let T be a boolean satisfiable $L_{\kappa\omega}$ -theory. Let $\Sigma(v)$ be an $L_{\kappa\omega}$ -type and assume that it is not isolated by an $L_{\kappa\omega}$ -sentence modulo T. Then there exists a full boolean valued model \mathcal{M} such that:

• $\mathcal{M} \vDash T$ and \mathcal{M} omits Σ .

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